

# Strong mixing and operator-selfdecomposability

Richard C. Bradley\* and Zbigniew J. Jurek<sup>†‡</sup>

*J. Theor. Probab.* **29**, 2016, pp. 292-306.

**Abstract.** For nonstationary, strongly mixing sequences of random variables taking their values in a finite-dimensional Euclidean space, with the partial sums being normalized via matrix multiplication, with certain standard conditions being met (an “infinitesimality” assumption, and a restriction to “full” distributions), the possible limit distributions are precisely the operator-selfdecomposable laws.

When one has observations (distributions) with values in an algebraic structure then their normalizations should be consistent with the structure in question. Thus, when  $\xi_1, \xi_2, \dots$  are  $\mathbb{R}^d$ -valued vectors then one should consider sums

$$A_n(\xi_1 + \xi_2 + \dots + \xi_n) + x_n, \quad (*)$$

where  $(A_n)$  are linear operators (matrices) on  $\mathbb{R}^d$ . Sharpe (1969), for independent identically distributed  $\xi_i$ 's in  $(*)$ , and Urbanik (1972), for infinitesimal triangular arrays  $(A_n \xi_i, 1 \leq i \leq n, n \geq 1)$ , described limit distributions in the scheme  $(*)$ . The monograph by Jurek and Mason (1993) summarized the research in that area for stochastically independent variables. The CLT for affine normalizations of independent, non-identically distributed random vectors, from Hahn and Klass (1981), still awaits a coordinate-free proof. For identically distributed summands, a coordinate-free proof can be found in Meerschaert and Scheffler (2001), Theorem 8.1.3. In fact, they characterized the generalized domain of attraction of a full normal law. Also cf. Jurek (1980).

---

\*Department of Mathematics, Indiana University Bloomington, Indiana, USA.

<sup>†</sup>Institute of Mathematics, University of Wrocław, Wrocław, Poland.

<sup>‡</sup>Research funded by Narodowe Centrum Nauki (NCN) grant no Dec2011/01/B/ST1/01257.

Here we will describe limiting distributions of  $(*)$  for  $\mathbb{R}^d$ -valued random variables  $\xi_1, \xi_2, \dots$ , that are only strongly mixing, as defined by Rosenblatt (1956). For the case of real-valued random variables, certain results on classical limiting distributions for strongly mixing sequences (normalized by scalars) are summarized in Bradley (2007), Chapter 12 in the case of strict stationarity, and by Bradley and Jurek (2014) (as a special case of the main result there) for the case of nonstationarity.

### 1. Strong mixing and operator-selfdecomposability.

Let  $\mathbb{R}^d$  be the  $d$ -dimensional Euclidean space. As in Jurek and Mason (1993), by  $\mathbf{End}(\mathbb{R}^d) \equiv \mathbf{End}$  we denote the Banach algebra of all bounded linear operators (matrices) on  $\mathbb{R}^d$  and by  $\mathbf{Aut}(\mathbb{R}^d) \equiv \mathbf{Aut}$  the group of all linear bounded and invertible operators (matrices). By  $\mathcal{P}(\mathbb{R}^d) \equiv \mathcal{P}$  we denote the topological semigroup of all Borel probability measures on  $\mathbb{R}^d$  with convolution  $*$  and weak convergence topology.

Furthermore, let  $(\Omega, \mathbb{F}, P)$  be a probability space rich enough to carry an uncountable family of independent uniformly distributed (on  $[0, 1]$ ) random variables as well as sequences  $\mathbf{X} := (X_1, X_2, \dots)$  of  $\mathbb{R}^d$ -valued random vectors (in short: random vectors); cf. Dudley (2002), Theorem 8.2.2.

We will say that a random vector  $X$  or its probability distribution  $\mu$  is *full or genuinely  $d$  dimensional* if its support is not contained in any proper hyperplane in  $\mathbb{R}^d$ . (Recall that a *hyperplane* is a linear subspace of  $\mathbb{R}^d$  shifted by a vector.) By  $\mathcal{F}$  we denote the family of all full probability measures on  $\mathbb{R}^d$ . It is an open (in weak convergence topology) subsemigroup of  $\mathcal{P}$  cf. Jurek-Mason (1993), Corollary 2.1.2.

With a random vector  $X$  or its probability distribution  $\mu$  we associate two semigroups of matrices: the *Urbanik decomposability semigroup*  $\mathbf{D}(X)$  (or  $\mathbf{D}(\mu)$ ) and the *symmetry semigroup*  $\mathbf{A}(X)$  (or  $\mathbf{A}(\mu)$ ) as follows:

$$\begin{aligned} \mathbf{D}(X) &:= \{A \in \mathbf{End} : X \stackrel{d}{=} AX + Y \text{ for some } Y \text{ independent of } X\}, \\ \mathbf{A}(X) &:= \{A \in \mathbf{End} : X \stackrel{d}{=} AX + a \text{ for some vector } a \in \mathbb{R}^d\}, \end{aligned} \quad (1)$$

where  $\stackrel{d}{=}$  denotes equality in distribution. In an analogous way we define semigroups  $\mathbf{D}(\mu)$  and  $\mathbf{A}(\mu)$ . Of course,  $\mathbf{A}(X) \subset \mathbf{D}(X)$  and the operators 0 (zero) and  $I$  (identity) are always in  $\mathbf{D}(X)$ .

(The symbol 0 will be used freely for the zero elements of  $\mathbb{R}$ ,  $\mathbb{R}^d$ , and  $\mathbf{End}$ . In context, that should not cause confusion.)

For the references below let us recall that

- (i)  $\mathbf{D}(\mu)$  is a compact semigroup in  $\mathbf{End}(\mathbb{R}^d)$  iff  $\mu$  is a full measure  
iff  $\mathbf{A}(\mu)$  is a compact group in  $\mathbf{Aut}(\mathbb{R}^d)$ ;
- (ii) If  $\mu$  is full, then  $\mathbf{A}(\mu)$  is the largest group in the Urbanik  
semigroup  $\mathbf{D}(\mu)$ . (2)

Cf. Jurek and Mason (1993), Theorem 2.3.1 and Corollary 2.3.2 and Proposition 2.3.4.

We will say that a probability measure  $\mu$  is operator-selfdecomposable if there exist a sequence  $b_n \in \mathbb{R}^d$ , a sequence  $A_n \in \mathbf{End}$  and a sequence  $X_n$  of independent  $\mathbb{R}^d$ -valued random vectors such that

- (i) the triangular array  $(A_n X_j : 1 \leq j \leq n, n \geq 1)$  is infinitesimal;
- (ii)  $A_n(X_1 + X_2 + \dots + X_n) + b_n \Rightarrow \mu$  as  $n \rightarrow \infty$ .

(Condition (i) simply means that  $A_n X_j$  converges uniformly to 0 in probability as  $n \rightarrow \infty$ ,  $j \in \{1, \dots, n\}$ .)

The main characterization due to K. Urbanik is as follows:

A full measure  $\mu$  is operator-selfdecomposable iff its decomposability semigroup  $\mathbf{D}(\mu)$  contains at least one one-parameter semigroup  $\{\exp(-tQ), t \geq 0\}$  with  $\exp(-tQ) \rightarrow 0$  (the zero matrix) as  $t \rightarrow \infty$ .

Cf. Jurek-Mason (1993), Theorem 3.3.5. (The stipulation in that theorem that  $Q$  be invertible, is superfluous; note that  $Q^{-1} = \int_0^\infty e^{-sQ} ds$ ; the integral is well defined because  $\exp(-tQ) \rightarrow 0$  as  $t \rightarrow \infty$ .)

Also it might be of some importance to mention here that we have the following random integral representation:

$$\mu \text{ is operator-selfdecomposable iff } \mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY(t)\right)$$

for some Lévy process  $Y$  (a so called *background driving Lévy process* (BDLP)); cf. Jurek (1982) or Jurek-Mason (1993), Theorem 3.6.6.

Since our aim here is to extend the notion of operator-selfdecomposability to some dependent random variables, let's recall that for two sub- $\sigma$ -fields  $\mathcal{A}$  and  $\mathcal{B}$  of  $\mathbb{F}$  we define the measure of dependence  $\alpha$  between them as follows:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

For a given sequence  $\mathbf{X} := (X_1, X_2, \dots)$  of  $\mathbb{R}^d$ -valued random variables, we define for each positive integer  $n$  the dependence coefficient

$$\alpha(n) \equiv \alpha(\mathbf{X}; n) := \sup_{j \in \mathbf{N}} \alpha(\sigma(X_k, 1 \leq k \leq j), \sigma(X_k, k \geq j+n)), \quad (3)$$

where  $\sigma(\dots)$  denotes the  $\sigma$ -field generated by  $(\dots)$ . We will say that a sequence  $\mathbf{X}$  is *strongly mixing* (Rosenblatt (1956)) if

$$\alpha(n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4)$$

Of course, if the elements of  $\mathbf{X}$  are stochastically independent then  $\alpha(\mathbf{X}; n) \equiv 0$ .

**THEOREM 1.** *Let  $\mathbf{X} := (X_1, X_2, \dots)$  be a sequence of  $\mathbb{R}^d$ -valued random vectors, with the partial sums  $S_n := X_1 + X_2 + \dots + X_n$ , and let  $(A_n) \in \mathbf{End}(\mathbb{R}^d)$  and  $(b_n) \in \mathbb{R}^d$  be sequences of bounded linear operators and vectors, respectively, satisfying conditions:*

- (i)  $\alpha(\mathbf{X}; n) \rightarrow 0$  as  $n \rightarrow \infty$ , i.e., the sequence  $\mathbf{X}$  is strongly mixing;
- (ii) the triangular array  $(A_n X_j, 1 \leq j \leq n, n \geq 1)$  is infinitesimal;
- (iii)  $A_n S_n + b_n \Rightarrow \mu$  as  $n \rightarrow \infty$ , for some full probability measure  $\mu$ .

*Then the limit distribution  $\mu$  is operator-selfdecomposable, that is, there exists a one parameter semigroup  $\{e^{-tQ} : t \geq 0\} \subset \mathbf{D}(\mu)$  with  $\lim_{t \rightarrow \infty} e^{-tQ} = 0$ .*

Note that every sequence of independent random variables is (trivially) strongly mixing; and hence by the exposition in Section 3.1 of Jurek-Mason (1993), every full operator-selfdecomposable law on  $\mathbb{R}^d$  can occur as the limit distribution  $\mu$  in Theorem 1.

The line of reasoning in our proof of this theorem is as follows. First, in Section 2 we investigate the normalizing sequence  $(A_n)$  of matrices, and in particular we show that one may choose a more appropriate sequence  $(\tilde{A}_n)$ . Then in Section 3, using the new normalizing sequence, we construct in a few steps a one-parameter semigroup  $(e^{-tQ}, t \geq 0)$ . Here, we follow Urbanik (1972). See also the remarks in Section 4.

## 2. Auxiliary propositions and lemmas.

First, some consequences of the operator-convergence of types theorems (Section 2.2 in Jurek-Mason (1993)):

**PROPOSITION 1.** *Under the assumptions (ii) and (iii) in Theorem 1,*

a)  $A_n \rightarrow 0$  as  $n \rightarrow \infty$ ; the inverse  $A_n^{-1}$  exists for all sufficiently large  $n$ ;

b) there exist  $\tilde{A}_n$  for which (ii) and (iii) (in Theorem 1) hold and

$\tilde{A}_{n+1} \tilde{A}_n^{-1} \rightarrow I$  (the identity matrix);

c) one has that

$$\lim_{n \rightarrow \infty} \left| \frac{\det A_{n+1}}{\det A_n} \right| = \lim_{n \rightarrow \infty} \frac{\det \tilde{A}_{n+1}}{\det \tilde{A}_n} = 1.$$

Cf. Jurek-Mason (1993), Section 3.2 : Propositions 3.2.1 and 3.2.2. For part c) one needs also Corollaries 2.3.2 and 2.4.2 as  $\tilde{A}_n := H_n A_n$  for some  $H_n \in \mathbf{A}(\mu)$ .

Second, a note on uniformly infinitesimal triangular arrays.

**LEMMA 1.** *Suppose that for each  $n \in \mathbb{N}$ ,  $I_n$  is a nonempty set. Suppose that for each  $n \in \mathbb{N}$  and each  $j \in I_n$ ,  $X_{n,j}$  is an  $\mathbb{R}^d$ -valued random element. The the following two statements are equivalent:*

$$(A) \quad \forall(\epsilon > 0) \quad \lim_{n \rightarrow \infty} \sup_{j \in I_n} P(\|X_{n,j}\| \geq \epsilon) = 0 \quad (5)$$

*There exists a sequence  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_{n-1} \geq \delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that*

$$(B) \quad \forall(n \in \mathbb{N}) \quad \forall(j \in I_n) \quad P(\|X_{n,j}\| \geq \delta_n) \leq \delta_n \quad (6)$$

*Proof.*  $((A) \Rightarrow (B))$  Let  $N_1 := 1$ . For already defined  $N_1, N_2, \dots, N_{m-1}$ , let  $N_m > N_{m-1}$  be such that

$$\forall(n \geq N_m) \quad \forall(j \in I_n) \quad P(\|X_{n,j}\| \geq 1/m) \leq \frac{1}{m} \quad (7)$$

which always exists by (5). Next, for the defined sequence

$$1 = N_1 < N_2 < \dots < N_n < \dots,$$

let us define the sequence  $(\delta_n)$  as follows: For each  $m \in \mathbb{N}$ ,

$$\delta_n := 1/m, \quad \text{for all } n \text{ such that } N_m \leq n \leq N_{m+1} - 1. \quad (8)$$

Thus by virtue of the above construction,  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ ; and for each  $n \in \mathbb{N}$  there exists exactly one  $m \in \mathbb{N}$  such that  $N_m \leq n \leq N_{m+1} - 1$ , and by (7) and (8),

$$P(\|X_{n,j}\| \geq \delta_n) = P(\|X_{n,j}\| \geq 1/m) \leq 1/m = \delta_n \quad \text{for all } j \in I_n,$$

which completes the proof  $(A) \Rightarrow (B)$ . The implication  $(B) \Rightarrow (A)$  is obvious.

**COROLLARY 1.** *If the infinitesimal triangular array  $(X_{n,j})$  is as in Lemma 1 and  $q_n \rightarrow \infty$ ,  $q_n \leq \delta_n^{-1/2}$ , then for any set  $Q \subset I_n$  such that  $\text{card } Q \leq q_n$  we have that*

$$P(\|\sum_{k \in Q} X_{n,k}\| \geq \delta_n^{1/2}) \leq \delta_n^{1/2}$$

Since  $\text{card } Q \leq q_n$  and  $\delta_n q_n \leq \delta_n^{1/2}$ ,

$$\begin{aligned} P(\|\sum_{k \in Q} X_{n,k}\| \geq \delta_n^{1/2}) &\leq P(\|\sum_{k \in Q} X_{n,k}\| \geq \delta_n q_n) \\ &\leq \sum_{k \in Q} P(\|X_{n,k}\| \geq \delta_n) \leq q_n \delta_n \leq \delta_n^{1/2}. \end{aligned}$$

Third, a generalization of Proposition 3.2.3 in Jurek-Mason, for strongly mixing sequences.

**PROPOSITION 2.** *Suppose the hypothesis of Theorem 1, including all of conditions (i), (ii), and (iii) there, hold. Suppose also that for every  $n \in \mathbb{N}$ , the matrix  $A_n$  is invertible. Then*

$$\sup\{\|A_n A_m^{-1}\| : n \in \mathbb{N}, 1 \leq m \leq n\} < \infty. \quad (9)$$

Moreover, if for each  $n \in \mathbb{N}$ ,  $m_n$  is an integer such that  $1 \leq m_n \leq n$ , then all limits points of the sequence  $(A_n A_{m_n}^{-1})_{n \in \mathbb{N}}$  are in  $\mathbf{D}(\mu)$ .

*Proof.* We shall first prove (9). Suppose that for each  $n \in \mathbb{N}$ ,  $m_n$  is an integer such that  $1 \leq m_n \leq n$ . To prove (9), it suffices to prove that

$$\sup_{n \in \mathbb{N}} \|A_n A_{m_n}^{-1}\| < \infty. \quad (10)$$

If instead  $\|A_n A_{m_n}^{-1}\| \rightarrow \infty$  along some subsequence of  $n \in \mathbb{N}$ , then within that subsequence the integers  $m_n$  could not be bounded (for otherwise  $\|A_n A_{m_n}^{-1}\| \rightarrow 0$  would occur along that subsequence by Proposition 1(a)), and there would be a further subsequence along which  $m_n \rightarrow \infty$ . Letting  $m_n := n$  for all  $n$  not in that “further subsequence,” we have reduced our task (for the proof of (9)) to proving (10) under the additional assumption that  $m_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

For the rest of the proof it is assumed that

$$m_n \rightarrow \infty, \quad 0 < q_n \rightarrow \infty \quad \text{and} \quad q_n \leq \delta_n^{-1/2},$$

where the sequence  $(\delta_n)_{n \in \mathbb{N}}$  is as in Lemma 1.

For  $n \in \mathbb{N}$ , using the notation  $S(\mathbf{X}, n) \equiv S_n \equiv X_1 + X_2 + \dots + X_n$ , define random vectors as follows:

$$\eta_n := \begin{cases} 0, & \text{if } m_n = n; \\ S(\mathbf{X}, n) - S(\mathbf{X}, m_n), & \text{if } n - q_n \leq m_n < n \\ S(\mathbf{X}, m_n + q_n) - S(\mathbf{X}, m_n), & \text{if } m_n \leq n - q_n - 1. \end{cases}$$

and

$$\xi_n := \begin{cases} 0, & \text{if } n - q_n \leq m_n \leq n \\ S(\mathbf{X}, n) - S(\mathbf{X}, m_n + q_n), & \text{if } m_n \leq n - q_n - 1. \end{cases}$$

Thus

$$S_n \equiv S(\mathbf{X}, n) = S(\mathbf{X}, m_n) + \eta_n + \xi_n \quad (11)$$

Since  $\eta_n$  is either zero or the sum of at most  $q_n \leq \delta_n^{-1/2}$  of the variables  $X_k$ , one has by Corollary 1 that  $A_n \eta_n \rightarrow 0$  in probability as  $n \rightarrow \infty$ . Consequently,

$$A_n S(\mathbf{X}, m_n) + A_n \xi_n + b_n \Rightarrow Z \quad \text{as } n \rightarrow \infty, \quad (12)$$

where  $Z$  is a random vector whose distribution is  $\mu$ . From the definition of the  $\xi_n$ 's, for the case  $m_n \leq n - q_n - 1$  we have

$$\alpha(\sigma(S(\mathbf{X}, m_n)), \sigma(\xi_n)) \leq \alpha(\mathbf{X}, q_n + 1) \quad (13)$$

In the opposite case ( $m_n \geq n - q_n$ ), the  $\xi_n$ 's are zero (constant variables) and the left-hand side of (13) is therefore zero. Now from (12),

$$[A_n A_{m_n}^{-1} (A_{m_n} S(\mathbf{X}, m_n) + b_{m_n})] + [A_n \xi_n + b_n - A_n A_{m_n}^{-1} b_{m_n}] \Rightarrow Z, n \rightarrow \infty \quad (14)$$

For simplicity, let  $V_n$  and  $W_n$  denote the first and the second expressions in the above square brackets, that is

$$V_n + W_n \Rightarrow Z \quad (15)$$

From (13) and Corollary 1.11 in Bradley (2007),

$$\begin{aligned} |\mathbb{E}[\exp i \langle t, V_n + W_n \rangle] - \mathbb{E}[\exp i \langle t, V_n \rangle] \cdot \mathbb{E}[\exp i \langle t, W_n \rangle]| \\ \leq 16 \alpha(\mathbf{X}, q_n + 1) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence by (15),

$$\mathbb{E}[\exp i \langle t, V_n \rangle] \cdot \mathbb{E}[\exp i \langle t, W_n \rangle] \rightarrow \mathbb{E}[\exp i \langle t, Z \rangle] \quad \text{as } n \rightarrow \infty.$$

Our next task is to replace vectors  $W_n$  by vectors that are stochastically independent of  $V_n$ . To this aim, let  $\zeta_{n,m}$ ,  $m = 1, 2, \dots, n$  be random vectors independent of  $\sigma(\mathbf{X}, Z)$  such that

$$\begin{aligned} \zeta_{n,m} &:= b_n - A_n A_m^{-1} b_m, \quad \text{if } m + q_n \geq n; \text{ (a constant); for } m + q_n < n, \\ \mathcal{L}(\zeta_{n,m}) &:= \mathcal{L}(A_n (S(\mathbf{X}, n) - S(\mathbf{X}, m + q_n)) + b_n - A_n A_m^{-1} b_m). \end{aligned}$$

But note that  $\mathcal{L}(W_n) = \mathcal{L}(\zeta_{n,m_n})$  and thus by the above,

$$\mathbb{E}[\exp i < t, V_n >] \cdot \mathbb{E}[\exp i < t, \zeta_{n,m_n} >] \rightarrow \mathbb{E}[\exp i < t, Z >] \text{ as } n \rightarrow \infty$$

and therefore

$$\mathcal{L}(V_n) * \mathcal{L}(\zeta_{n,m_n}) = \mathcal{L}(V_n + \zeta_{n,m_n}) \Rightarrow Z$$

By Parthasarathy (1967), Theorem 2.2 in Chapter III,  $(\mathcal{L}(V_n))_n$  is shift compact, or in the “symmetrization” terminology there,

$$(A_n A_{m_n}^{-1} (A_{m_n} S(\mathbf{X}^\circ, m_n)))_{n \in \mathbb{N}}, \text{ is compact and } A_{m_n} S(\mathbf{X}^\circ, m_n) \Rightarrow Z^\circ.$$

From Lemma 2.2.3 in Jurek-Mason (1993) we now get the boundedness in Proposition 2.

Further, if  $D$  is a limit point of the family of matrices  $(A_n A_{m_n}^{-1})_{n \in \mathbb{N}}$  then from (14) we get  $\mathcal{L}(DZ + Y) = \mathcal{L}(Z)$  for some random variable  $Y$ , a limit point of  $(\zeta_{n,m_n})_{n \in \mathbb{N}}$ , independent of  $Z$ . This completes the proof Proposition 2.

### 3. Construction of the one-parameter semigroup in $\mathbf{D}(\mu)$ .

Here we follow the Urbanik construction from Urbanik (1972). (See also the remarks in Section 4.) Throughout this section, as in the hypothesis of Theorem 1, we assume that *the probability measure  $\mu$  is full*; and as allowed by Proposition 1, we assume that the matrices  $A_n$  satisfying conditions (ii) and (iii) in Theorem 1 are invertible and satisfy  $A_{n+1} A_n^{-1} \rightarrow I$  and (hence)  $\det A_{n+1} / \det A_n \rightarrow 1$  (as  $n \rightarrow \infty$ ).

By an *idempotent*  $J$ , in  $\mathbf{End}(\mathbb{R}^d)$ , we mean a projector from  $\mathbb{R}^d$  onto the linear subspace  $J(\mathbb{R}^d)$ , that is  $J^2 = J$ . Following Numakura (1952), p. 103, we will say that *idempotent  $K$  is under idempotent  $J$* , if  $K \neq J$  and  $JK = KJ = K$ . Hence, in particular,  $K(\mathbb{R}^d) \subsetneq J(\mathbb{R}^d)$ . If there is no non-zero idempotent under  $J$ , then we will say that  $J$  is a *primitive idempotent*.

Idempotents will play a crucial role below as we have the following: *for an idempotent  $J$  we have that*

$$J \in \mathbf{D}(\mu) \text{ iff } (I - J) \in \mathbf{D}(\mu) \text{ and } \mu = J\mu * (I - J)\mu$$

Furthermore, if an idempotent  $K$  is under  $J$  and both are in  $\mathbf{D}(\mu)$  then

$$\mu = K\mu * (J - K)\mu * (I - J)\mu \text{ and } K + (J - K) + (I - J) = I \quad (16)$$

For details cf. Jurek-Mason (1993), Theorem 2.3.6.



Below,  $\det_J A$  means the determinant of the matrix representation of the operator  $JA$  in  $J(\mathbb{R}^d)$  relative to an orthogonal basis of  $J(\mathbb{R}^d)$ . Hence we get

$$\begin{aligned} (a) \quad & \det_J A = \det_J(JA) = \det_J(AJ) = \det_J(JAJ); \\ (b) \quad & \det_J(AJB) = \det_J A \det_J B; \\ (c) \quad & \det(JAJ + (I - J)B(I - J)) = \det_J A \det_{I-J} B. \end{aligned} \quad (17)$$

**LEMMA 2.** *For a given idempotent  $J \in \mathbf{D}(\mu)$ , for each  $0 < c < 1$  there exist  $K_c \in \mathbf{D}(\mu)$  such that  $\det_J K_c = c$*

*Proof.* For  $1 \leq n \leq m$ , one has the inequalities

$$\begin{aligned} \|A_{m+1}A_n^{-1} - A_m A_n^{-1}\| &\leq \|A_m A_n^{-1}\| \|A_{m+1}A_m^{-1} - I\| \\ &\leq \left(\sup_{n \leq m} \|A_m A_n^{-1}\|\right) \|A_{m+1}A_m^{-1} - I\|. \end{aligned}$$

Since by Proposition 2,  $\sup\{\|A_n A_m^{-1}\| : 1 \leq m \leq n, n \in \mathbb{N}\} < \infty$ , we get

$$\lim_{m \rightarrow \infty} \sup_{n \leq m} \|\|A_{m+1}A_n^{-1} - A_m A_n^{-1}\|\| \leq \lim_{m \rightarrow \infty} \sup_{n \leq m} \|A_{m+1}A_m^{-1} - I\| = 0.$$

Since the functions  $\mathbb{R}^{d^2} \ni A \rightarrow \|A\|$  and  $\mathbb{R}^{d^2} \ni A \rightarrow \det_J A$  are continuous, therefore by putting  $b_{m,n} := \det_J A_m A_n^{-1}$  ( $n \leq m$ ) we infer that

$$b_{n,n} = 1, \quad \lim_{m \rightarrow \infty} b_{m,n} = 0 \quad (n = 1, 2, \dots), \quad \lim_{m \rightarrow \infty} \sup_{n \leq m} |b_{m+1,n} - b_{m,n}| = 0. \quad (18)$$

Thus for any  $0 < c < 1$  and the sequence  $m_n := \sup\{k \geq n : b_{k,n} \geq c\}$  we get  $b_{m_n+1,n} < c \leq b_{m_n,n}$ , so from (18),  $\lim_{n \rightarrow \infty} b_{m_n,n} = c$ .

Furthermore, by Proposition 2, if  $K_c$  is a limit point of a sequence  $(A_{m_n} A_n^{-1})$  then  $K_c$  is in  $\mathbf{D}(\mu)$  and, by (18),  $\det_J K_c = c$ , which concludes the proof.

**LEMMA 3.** *Let  $J$  be non-zero idempotent in  $\mathbf{D}(\mu)$ . Then there exists  $T_n \in \mathbf{D}(\mu)$ ,  $n = 1, 2, \dots$  such that*

$$JT_n = T_n J = T_n, \quad T_n \rightarrow J \text{ and } \lim_{k \rightarrow \infty} T_n^k = 0 \quad (n = 1, 2, \dots) \quad (19)$$

*Proof.* We shall justify the above claim by the mathematical induction with respect to the dimension of the linear space  $J(\mathbb{R}^d)$ .

Step 1.  $\dim J(\mathbb{R}^d) = 1$ .

From Lemma 2, there exist  $K_n \in \mathbf{D}(\mu)$  such that  $\det_J K_n = 1 - 1/n$ . Putting  $T_n := JK_n J$  we have that the linear transformation  $T_n : J(\mathbb{R}^d) \rightarrow$

$J(\mathbb{R}^d)$  must be a multiple of  $J$ ; ( $\dim J(\mathbb{R}^d) = 1$ ). But  $\det_J T_n = \det_J K_n = 1 - 1/n$  and thus  $T_n = (1 - 1/n)J$  which satisfies (19).

Step 2. Assume  $\dim J(\mathbb{R}^d) = l > 1$  and for all idempotents  $K \in \mathbf{D}(\mu)$  such that  $\dim K(\mathbb{R}^d) < l$ , Lemma 3 is true.

Case (i). Assume that there exists a non-zero idempotent  $L \in \mathbf{D}(\mu)$  such that  $L \neq J$  and

$$L = JL = LJ, \quad (20)$$

that is, the idempotent  $J$  is not a primitive one.

From the above,  $J - L$  is also an idempotent. From Jurek-Mason (1993) Theorem 2.3.6 (a),  $I - L \in \mathbf{D}(\mu)$ . Hence  $J(I - L) = J - L \in \mathbf{D}(\mu)$ .

Since  $\dim L(\mathbb{R}^d) < l$  and  $\dim(J - L)(\mathbb{R}^d) < l$  therefore, by the mathematical induction assumption, there exist sequences  $(U_n)$  and  $(V_n)$  in  $\mathbf{D}(\mu)$  such that

$$\begin{aligned} U_n &\rightarrow L, \quad LU_n = U_n L = U_n \quad \text{and} \quad \lim_{k \rightarrow \infty} U_n^k = 0 \quad (n = 1, 2, \dots), \\ V_n &\rightarrow J - L, \quad (J - L)V_n = V_n(J - L) = V_n \quad \text{and} \quad \lim_{k \rightarrow \infty} V_n^k = 0 \quad (n = 1, 2, \dots) \end{aligned}$$

Then putting  $T_n := U_n + V_n$  we have  $T_n \rightarrow J$ . Further, from the identity  $T_n = LU_n L + (I - L)V_n(I - L)$  and again by Theorem 2.3.6 (d) in Jurek-Mason (1993) we get that  $T_n \in \mathbf{D}(\mu)$  and also  $T_n^k = U_n^k + V_n^k \rightarrow 0$  as  $k \rightarrow \infty$ . (Also, the first two equalities in (19) hold by an elementary argument.) This completes the Case (i).

Case(ii). There are no non-zero idempotents  $L$  in  $\mathbf{D}(\mu)$  different from  $J$  and satisfying  $JL = LJ = L$ , i.e., idempotent  $J$  is a primitive idempotent.

From Lemma 2, choose  $D_n \in \mathbf{D}(\mu)$  such that

$$0 < \det_J D_n < 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \det_J D_n = 1. \quad (21)$$

By (a) in the formula (17) we may assume that  $JD_n = D_n J = D_n$  and if  $D$  is a limit point of the sequence  $D_n$  then we also have equalities

$$D \in \mathbf{D}(\mu), \quad JD = DJ = D \quad \text{and} \quad \det_J D = 1 \quad (22)$$

Put  $A := D + I - J$ . Note that  $A = JDJ + (I - J)I(I - J)$ . Then by (d) in Theorem 2.3.6 from Jurek-Mason (1993),  $A \in \mathbf{D}(\mu)$ . However, by (17) and (21),

$$\det A = \det_{J+I-J}(JDJ + (I - J)I(I - J)) = \det_J D \det_{I-J}(I - J) = 1.$$

Consequently, by Jurek-Mason (1993), Proposition 2.3.5 and Corollary 2.3.2,

$$A \in \mathbf{A}(\mu) \text{ (a compact group in } \mathbf{Aut} \text{) and } A^{r_n} \rightarrow I, \quad (23)$$

for some  $r_1 < r_2 < \dots$ . Since  $JA^n = D^n$  we have that  $D^{r_n} \rightarrow J$ . Furthermore, since  $D$  is a limit point of the sequence  $D_n$  we can choose a subsequence  $(k_n)$  such that

$$T_n := D_{k_n}^{r_n} \rightarrow J; \quad JT_n = T_n J = T_n; \quad 0 < \det_J T_n < 1.$$

To complete the proof one needs to show that  $T_n^k \rightarrow 0$  as  $k \rightarrow \infty$ .

For each  $n$ , the monothetic semigroup  $\text{sem}(T_n)$  (the smallest closed sub-semigroup containing  $T_n$ ) is compact in  $\mathbf{D}(\mu)$ . By the Numakura Theorem (Corollary 1.1.3 in Jurek-Mason) the limit points of  $(T_n^k)_{k \in \mathbb{N}}$  form a group, denoted by  $K(T_n)$ , with the unit  $L$  that satisfies

$$JL = LJ = L \quad \text{and} \quad \det_J L = 0 \quad \text{and thus} \quad L \neq J$$

Because of the assumption (ii) we must have  $L = 0$ . Consequently  $T_n^k \rightarrow 0$  as  $k \rightarrow \infty$ , which completes the proof of Lemma 3.

Using the formula (16) inductively, there are finitely many non-zero primitive idempotents  $J_1, J_2, \dots, J_q$  in  $\mathbf{D}(\mu)$ ,  $q \leq d$  (the dimension of  $\mathbb{R}^d$ ), such that

$$I = J_1 + J_2 + \dots + J_q, \quad J_r J_s = J_s J_r = 0 \quad (1 \leq r \neq s \leq q). \quad (24)$$

Thus, in particular, for every  $s$  there is no non-zero idempotent  $K$  such that

$$J_s K = K J_s = K. \quad (25)$$

Finally, recall that for idempotents satisfying (24) (not necessary primitive ones) we have

$$\text{if } A_1, A_2, \dots, A_q \in \mathbf{D}(\mu) \text{ then } J_1 A_1 J_1 + J_2 A_2 J_2 + \dots + J_q A_q J_q \in \mathbf{D}(\mu); \quad (26)$$

for details cf. Jurek-Mason (1993), Theorem 2.3.6.

**LEMMA 4.** *There exists a positive integer  $q$  and a one parameter semigroup  $\{C_w : w \in W\} \subset \mathbf{D}(\mu)$  (where  $W$  denotes the set of non-negative rational numbers) such that  $\det C_w = e^{-qw}$  and  $C_0 = I$ .*

*Proof.* In view of Lemma 3, for the idempotents  $J_r$  in (24), let us choose  $T_{n,r} \in \mathbf{D}(\mu)$  such that for  $1 \leq r \leq q, n \geq 1$  we have

$$J_r T_{n,r} = T_{n,r} J_r = T_{n,r}, \quad T_{n,r} \rightarrow J_r, \quad \lim_{k \rightarrow \infty} T_{n,r}^k = 0, \quad 0 < \det_{J_r} T_{n,r} < 1. \quad (27)$$

Note that  $\lim_{n \rightarrow \infty} (\log \det_{J_r} T_{n,r}) = 0$  and put  $d(n, r) := [(-\log \det_{J_r} T_{n,r})^{-1}]$ , where the bracket  $[.]$  denotes the integer part. Hence,  $\lim_{n \rightarrow \infty} d(n, r) = \infty$  and

$$\lim_{n \rightarrow \infty} (d(n, r) \cdot (-\log \det_{J_r} T_{n,r})) = 1, \quad (r = 1, 2, \dots, q). \quad (28)$$

Further, let  $W$  denote the set of all non-negative rational numbers (as in the statement of Lemma 4). Then

$$T_{n,r}^{[w d(n,r)]} \in \mathbf{D}(\mu) \text{ for all } n \in \mathbb{N}, w \in W, 1 \leq r \leq q; \text{ and by (26),}$$

$$\sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \in \mathbf{D}(\mu) \quad \text{for all } w \in W, \quad n \in \mathbb{N}; \quad (29)$$

Since  $\mathbf{D}(\mu)$  is compact, there exist a subsequence  $Q \subset \mathbb{N}$  and  $C_w \in \mathbf{D}(\mu)$  such that for each  $w \in W$

$$\sum_{r=1}^q T_{n,r}^{[w d(n,r)]} = \sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \rightarrow C_w, \text{ as } n \rightarrow \infty, n \in Q. \quad (30)$$

(Note for  $w = 0$  that this gives  $C_0 = I$  by (24).) Hence, from (28) and (30) we get

$$\det_{J_s} C_w = \lim_{n \rightarrow \infty, n \in Q} \det_{J_s} T_{n,s}^{[w d(n,s)]} = \lim_{n \rightarrow \infty, n \in Q} (\det_{J_s} T_{n,s})^{[w d(n,s)]} = e^{-w}. \quad (31)$$

So, by (17) and (24) we conclude

$$\begin{aligned} \det C_w &= \lim_{n \rightarrow \infty} \det_{J_1 + \dots + J_q} \left( \sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \right) \\ &= \prod_{r=1}^q \lim_{n \rightarrow \infty, n \in Q} \det_{J_r} T_{n,r}^{[w d(n,r)]} = e^{-qw} \end{aligned} \quad (32)$$

To show that  $\{C_w : w \in W\}$  is indeed a one-parameter additive semigroup, note that for the integer part function  $a \ni \mathbb{R} \rightarrow [a] \in \mathbb{Z}$  (integers) we have

$$[a + b] - [a] - [b] \in \{0, 1\} \quad (33)$$

(because  $a + b - 1 < [a + b] \leq a + b$ ,  $-a \leq -[a] < 1 - a$  and  $-b \leq -[b] < 1 - b$ )

Hence, for  $w \in W$  and  $u \in W$ ,

$$s_n := [(w + u) d(n, r)] - [w d(n, r)] - [u d(n, r)] \in \{0, 1\}, \quad (r = 1, 2, \dots, q)$$

Hence by (27),

$$\lim_{n \rightarrow \infty} J_r (T_{n,r}^{s_n} - I) J_r = 0, \quad (r = 1, 2, \dots, q), \quad (34)$$

since  $s_n = 0$  or  $s_n = 1$ . Finally, from (30) and (24),

$$\begin{aligned} C_{w+u} - C_w C_u &= \\ \lim_{n \rightarrow \infty} \sum_{r=1}^q J_r T_{n,r}^{[(w+u)d(n,r)]} J_r - \left( \lim_{n \rightarrow \infty} \sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \right) \left( \lim_{n \rightarrow \infty} \sum_{s=1}^q J_s T_{n,s}^{[u d(n,s)]} J_s \right) \\ &= \sum_{r=1}^q \lim_{n \rightarrow \infty} T_{n,r}^{[w d(n,r)] + [u d(n,r)]} J_r (T_{n,r}^{s_n} - I) J_r. \end{aligned}$$

Since  $T_{n,r}^k \in \mathbf{D}(\mu)$  ( $n, k \in \mathbb{N}, r = 1, 2, \dots, q$ ) and  $\mathbf{D}(\mu)$  is compact (thus the norms of its members are bounded, say by  $B$ ), one has from above and (34),

$$\|C_{w+u} - C_w C_u\| \leq B \sum_{r=1}^q \lim_{n \rightarrow \infty} \|J_r (T_{n,r}^{s_n} - I) J_r\| = 0,$$

which gives the one-parameter semigroup property  $C_{w+u} = C_w C_u$ .

**LEMMA 5.** *For the given (full) probability measure  $\mu$ , its Urbanik decomposability semigroup  $\mathbf{D}(\mu)$  contains at least one one-parameter semigroup  $\{e^{-tQ}, t \geq 0\}$  ( $Q$  is a matrix) such that  $e^{-tQ} \rightarrow 0$ , as  $t \rightarrow \infty$ .*

*Proof.* Throughout this proof, we use freely all notations and arguments in the *proof* (as well as the statement) of Lemma 4.

Step 1. Let  $\mathbf{S} := \overline{\{C_w : w \in W\}}$  (the closure in  $\mathbf{Aut}$ ). Then  $\mathbf{S}$  is a compact semigroup in  $\mathbf{D}(\mu)$ . Further, since  $\det C_w = e^{-qw}$ , therefore it is an invertible operator. Thus

$$\mathbf{H} := \{C_w : w \in W\} \cup \{C_w^{-1} : w \in W\} \text{ is a commutative group in } \mathbf{Aut}.$$

To this end we have check that for  $w, u \in W$ , both  $C_w C_u^{-1}$  and  $C_w^{-1} C_u$  are in  $\mathbf{H}$ . Let assume that  $w > u$  then  $C_w C_u^{-1} = C_{w-u} C_u C_u^{-1} = C_{w-u} \in \mathbf{H}$ . Similarly,  $C_w^{-1} C_u = (C_u C_{w-u})^{-1} C_u = C_{w-u}^{-1} C_u^{-1} C_u = C_{w-u}^{-1} \in \mathbf{H}$ . (These equations yield both closure and, with a trivial extra argument, commutativity.)

Step 2. Let  $\mathbf{G} := \mathbf{S} \cup \mathbf{S}^{-1}$ . Then  $\mathbf{G} \subset \mathbf{Aut}$  is a commutative compactly generated subgroup. Moreover, the mapping  $h : \mathbf{G} \rightarrow (\mathbb{R}, +)$  given by  $h(A) := \log \det A$  is a homomorphism of those two topological groups with the kernel  $\ker h = \mathbf{S}_0 := \mathbf{S} \cap \mathbf{A}(\mu)$ . Thus the quotient group  $\mathbf{G}/\ker h$  is isomorphic with  $(\mathbb{R}, +)$ .

To see the above claim, first of all note that since  $\mathbf{S}_0$  is a closed sub-semigroup in the compact group  $\mathbf{A}(\mu)$ , therefore  $\mathbf{S}_0$  is a compact group,

by Theorem 1.1.12 in Paalman - de Miranda (1964) (see Theorem 2 in the Appendix).

If  $A \in \mathbf{S}_0$  then  $A \in \mathbf{A}(\mu)$  and by Corollaries 2.3.2 and 2.4.2 from Jurek and Mason (1993), we have that  $|\det A| = 1$ . On the other hand, since  $A \in \mathbf{S}$  we have that  $0 < \det A \leq 1$ , so  $\det A = 1$  and  $h(A) = 0$  and  $\mathbf{S}_0 \subset \ker h$ .

Conversely, if  $\det A = 1$  and  $A \in \mathbf{S}$  then  $A \in \mathbf{D}(\mu)$  and by Jurek-Mason (1993), Proposition 2.3.5 we get that  $A \in \mathbf{A}(\mu)$ . Consequently,  $A \in \mathbf{S}_0$ . If  $\det A = 1$  and  $A \in \mathbf{S}^{-1}$  then  $A^{-1} \in \mathbf{S}$  and  $\det A^{-1} = 1$  so  $A \in \mathbf{S}_0$ . This completes the proof of the Step 2.

Step 3. There is an isomorphism  $g : \mathbf{G} \rightarrow \mathbb{R} \oplus \mathbf{S}_0$  between the two topological groups.

This is so, because  $\mathbf{G}$  is a commutative and compactly generated group and the Pontriagin Theorem, from Montgomery and Zippin (1955), p. 187 (see Theorem 4 in the Appendix), gives the needed isomorphism.

Step 4. Taking the unit  $\mathcal{I}$  in the group in  $\mathbf{S}_0$  and putting for  $t \geq 0$ ,

$$\begin{aligned} T_t &:= g^{-1}(-t \oplus \mathcal{I}), \quad \text{if } g(\mathbf{S}) = (-\infty, 0] \oplus \mathbf{S}_0 \\ T_t &:= g^{-1}(t \oplus \mathcal{I}), \quad \text{if } g(\mathbf{S}) = [0, \infty) \oplus \mathbf{S}_0 \end{aligned} \quad (35)$$

we obtain the one-parameter semigroup of matrices in  $\mathbf{D}(\mu)$ .

From the equality  $g(\mathbf{G}) = g(\mathbf{S}) \cup (g(\mathbf{S}))^{-1}$ , and the fact  $g(\mathbf{S})$  is closed subsemigroup we infer that either  $g(\mathbf{S}) = (-\infty, 0] \oplus \mathbf{S}_0$  or  $g(\mathbf{S}) = [0, \infty) \oplus \mathbf{S}_0$ .

Step 5. For  $t \geq 0$ ,  $T_t = \exp(-tV)$  for some matrix  $V$ , and  $T_t \rightarrow 0$  as  $t \rightarrow \infty$ .

By Hille (1948), Theorem 8.4.2 (or Hille and Phillips (1957), Theorem 9.4.2 — see Theorem 3 in the Appendix — with the idempotent there being the identity matrix here in our context), we get the exponential form, that is,  $T_t = \exp tQ, t \geq 0$ , for some matrix  $Q$ .

For  $t > 0$  we have that  $T_t \notin \mathbf{S}_0$  and thus

$$0 < \det T_t < 1 \quad \text{for all } t > 0 \quad (36)$$

From the definitions of operators  $C_w, T_{n,r}$  and semigroup  $\mathbf{S}$  it follows that the idempotents  $J_r, 1 \leq r \leq q$  commute with  $\mathbf{S}$

Since  $T_t \in \mathbf{D}(\mu), t \geq 0$ , the set  $\{T_t, t \geq 0\}$  is conditionally compact. Hence by the Numakura Theorem, among the limits points (as  $t \rightarrow \infty$ ) there is an idempotent, say  $K$ . Of course by (36) and a simple argument,  $\det K = 0$ ; and by (24),  $K = J_1 K + \dots + J_q K$ . Also,  $K$  is the limit of a sequence of  $C_w$ 's with  $w \rightarrow \infty$  (forced by (32) since  $\det K = 0$ ), and hence by (31),

$$\det_{J_r} K = 0 \quad (r = 1, 2, \dots, q)$$

Since  $K$  and  $J_r$  commute and both are idempotents then so is  $J_r K$ . From above and (17),  $\det_{J_r} J_r K = \det_{J_r} K = 0$ , so  $J_r \neq J_r K$ . Moreover, we also have that  $J_r(J_r K) = (J_r K)J_r = J_r K$ . Thus from the properties of  $J_r$  ((24) and the *entire sentence* containing (25)) we must have  $J_r K = 0$  and consequently  $K = 0$ . That is, the only limit point of  $T_t$  as  $t \rightarrow \infty$ . As a consequence, Lemma 5 holds.

*Proof of Theorem 1.* It follows from Lemma 5.

#### 4. Concluding Remarks.

1). The construction of a one-parameter semigroup in the Urbanik decomposability semigroup  $\mathbf{D}(\mu)$ , given in Section 3, is based on finite-dimensional arguments. However, the construction in Urbanik (1978) and in Jurek and Mason (1993), Chapter 3, are valid for infinite dimensional Banach spaces.

2). The problem of limiting distributions in (\*) has a natural generalization for Banach-valued random variables  $\xi_j$  and bounded linear operators  $A_n$ . However, in that case one needs an additional technical assumption; cf. Urbanik (1978). (For the case of Banach-valued random variables  $\xi_j$  and scalars  $A_n$ , see Bradley and Jurek (2014).)

3) Similarly, in (\*) one may consider topological-group-valued random variables and endomorphisms of the group in question. To our knowledge this is an open question.

**5. Appendix.** For ease of reference let us quote here the following algebraic facts.

**THEOREM 2.** *Each locally compact subsemigroup  $S$  of a compact group  $G$  is a compact subgroup.*

Cf. A.B. Paalman - De Miranda (1964), Theorem 1.1.12.

**THEOREM 3.** *If  $T : (0, \infty) \rightarrow \mathfrak{B}$  (a real or complex Banach algebra) satisfies*

$$T(t+s) = T(t)T(s) \text{ for all } 0 < t, s < \infty \text{ and } \lim_{t \rightarrow 0} T(t) = J \text{ (an idempotent),}$$

*then there exists an element  $A \in \mathfrak{B}$  such that*

$$T(t) = J + \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \quad (\text{absolutely convergent series}). \quad (37)$$

Cf. E. Hille (1948), Theorem 8.4.2 or E. Hille and R. Phillips (1957), Theorem 9.4.2.

**THEOREM 4.** (Pontriagin Theorem) *Suppose a topological group  $G'$ , generated by a compact set, contains a compact subgroup  $H'$  such that  $G'/H'$  is isomorphic with an  $r$ -dimensional real vector group  $V_r$ . Then  $G'$  has a vector subgroup  $E_r$  such that  $G' = H' \oplus E_r$*

Cf. Montgomery and Zippin (1955), p. 187.

**Acknowledgement.** The authors thank the referee and the associate editor for their helpful comments, which improved the exposition.

## REFERENCES

- [1] A. Araujo and E. Gine (1980), *The central limit theorem for real and Banach valued random variables*, John Wiley & Sons, New York.
- [2] P. Billingsley (1999), *Convergence of probability measures*, Second edition, John Wiley & Sons, New York.
- [3] R.C. Bradley (2007), *Introduction to strong mixing conditions*, vol. 1, Kendrick Press, Heber City (Utah).
- [4] R.C. Bradley and Z.J. Jurek (2014), The strong mixing and the selfdecomposability properties, *Statist. Probab. Letters* **84**, pp. 67-71.
- [5] R.M. Dudley (2002), *Real analysis and probability*, Second edition, Cambridge University Press, Cambridge.
- [6] M.G. Hahn and M. Klass (1981), The multidimensional central limit theorem for arrays normed by affine transformations, *Ann. Probab.* **9**, pp. 611-623.
- [7] E. Hille (1948), *Functional analysis and semi-groups*, American Mathematical Society, New York.
- [8] E. Hille and R.S. Phillips (1957), *Functional analysis and semi-groups*, American Mathematical Society, Providence.
- [9] Z.J. Jurek (1980), Central limit theorem in Euclidean spaces, *Bull. Acad. Polon. Sci.* **28**, pp. 81-86.
- [10] Z.J. Jurek (1982), An integral representation of operator-selfdecomposable random variables, *Bull. Acad. Polon. Sci.* **30**, pp. 385-393.
- [11] Z.J. Jurek and J. David Mason (1993), *Operator-limit distributions in probability theory*, John Wiley & Sons, New York.
- [12] M.M. Meerschaert and H.P. Scheffler (2001), *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*, Wiley Interscience, John Wiley & Sons, New York.
- [13] D. Montgomery and L. Zippin (1955), *Topological transformation groups*, Interscience Publishers, New York and London.



- [14] K. Numakura (1952), On bicomact semigroups, *Math. J. Okayama University*, **1**, pp. 99-108.
- [15] A.B. Paalman-De Miranda (1964), *Topological semigroups*, Mathematisch Centrum, Amsterdam.
- [16] K.R. Parthasarathy (1967), *Probability measures on metric spaces*, Academic Press, New York and London.
- [17] M. Rosenblatt (1956), A central limit theorem and a strong mixing, *Proc. Natl. Acad. Sci. USA* **42**, pp. 43-47.
- [18] M. Sharpe (1969), Operator-stable probability measures on vector groups, *Trans. Amer. Math. Soc.* **136**, pp. 51-65.
- [19] K. Urbanik (1972), Lévy's probability measures on Euclidean spaces, *Studia Math.* **44**, pp. 119-148.
- [20] K. Urbanik (1978), Lévy's probability measures on Banach spaces, *Studia Math.* **63**, pp. 283-308.